STAT 131 — Discussion (Lecture 9 & 10) Solutions

Prepared for students

University of California, Santa Cruz

How to use this sheet

For each problem, (1) identify the right object (PMF/PDF/CDF); (2) set up the event/region carefully (draw when helpful); (3) compute with a short, clean integral or sum; (4) sanity-check units, signs, and that probabilities are in $[0, 1]$.

Lecture 9 — Joint PMF practice

Let X be the number of cars and Y the number of TVs for a randomly chosen household. The joint PMF is:

	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$
$X = 1$	0.10	0	0.10	0
$X = 2$	0.30	0	0.10	0.20
$X = 3$	0	0.20	0	0

(Quick check: totals by row $= 0.20, 0.60, 0.20$; grand total $= 1$.)

(a) $P(X > 2,\ Y > 2)$

We need entries with $X = 3$ and $Y \in \{3, 4\}$. From the table: $(3, 3) = 0$ and $(3, 4) = 0$.

$$\boxed{0.}$$

(b) $P(X = Y)$

Sum the diagonal $x = y$: $(1, 1) = 0.10$, $(2, 2) = 0$, $(3, 3) = 0$.

$$\boxed{0.10.}$$

(c) $P(X = 1)$

Row sum for $X = 1$: $0.10 + 0 + 0.10 + 0 = 0.20$.

$$\boxed{0.20.}$$

Lecture 10 — Independent continuous r.v.'s

Let $g(x) = \dfrac{3}{8}x^2$ on $0 < x < 2$ (and 0 otherwise). Suppose X and Y are independent and both have PDF g.

(a) Joint PDF of (X, Y)

Independence $\Rightarrow$ $f_{X,Y}(x, y) = g(x)\, g(y)$ on the product support:

$$\boxed{f_{X,Y}(x, y) = \left(\frac{3}{8}x^2\right)\left(\frac{3}{8}y^2\right) = \frac{9}{64}\, x^2y^2, \quad 0 < x < 2,\ 0 < y < 2 \text{ (and 0 otherwise).}}$$

(b) $P(X = Y)$

For continuous distributions without point masses, single points/curves have probability 0. Here the event $\{X = Y\}$ is the diagonal line in $\mathbb{R}^2$, which has 2D area 0.

$$\boxed{0.}$$

Reason: If X, Y have a joint PDF, then for any C^1 curve $\mathcal{C}$ in the plane, $P\big((X, Y) \in \mathcal{C}\big) = \int_{\mathcal{C}} f_{X,Y}\ \mathrm{d}s = 0$ (a one-dimensional set in $\mathbb{R}^2$ has Lebesgue measure 0).

(c) $P(X > Y)$

By symmetry for i.i.d. continuous variables, $P(X > Y) = P(Y > X)$ and $P(X = Y) = 0$, so

$$P(X > Y) = \frac{1 - P(X = Y)}{2} = \frac{1}{2}.$$

$$\boxed{\tfrac{1}{2}.}$$

(Optional direct check) $P(X > Y) = \int_0^2 \int_0^x \frac{9}{64}x^2y^2\ \mathrm{d}y\ \mathrm{d}x = \frac{1}{2}$.

(d) $P(X + Y \le 1)$

On $0 < x < 2$, $0 < y < 2$, the region $x + y \le 1$ restricts to the triangle $0 < x < 1$, $0 < y < 1 - x$:

$$P(X+Y \le 1) = \int_{x=0}^{1} \int_{y=0}^{1-x} \frac{9}{64}x^2y^2\ \mathrm{d}y\ \mathrm{d}x = \frac{9}{64}\int_0^1 x^2 \left[\frac{(1-x)^3}{3}\right] \mathrm{d}x = \frac{3}{64}\int_0^1 x^2(1-x)^3\ \mathrm{d}x.$$

Use the Beta integral $\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \mathrm{B}(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Here $a = 3$, $b = 4$, so $\int_0^1 x^2(1-x)^3\,\mathrm{d}x = \mathrm{B}(3,4) = \dfrac{2!\,3!}{6!} = \dfrac{1}{60}$.

$$P(X+Y \le 1) = \frac{3}{64}\cdot\frac{1}{60} = \boxed{\frac{1}{1280} \approx 7.81 \times 10^{-4}.}$$

Why so small?

$g(x) \propto x^2$ concentrates near 2, so mass near 0 is tiny. The chance both X and Y are simultaneously small enough to keep $X+Y \le 1$ is therefore very small.