STAT 131 — Discussion (Lecture 9) Solutions

Prepared for students

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What we do in this sheet

- Start from a joint pdf on a triangle.
- Fix the normalization first (make it integrate to 1).
- Build the joint CDF by cases (below the line $x + y = 1$ and above it).
- Differentiate the CDF to check we get back the pdf.
- Read off the marginals.

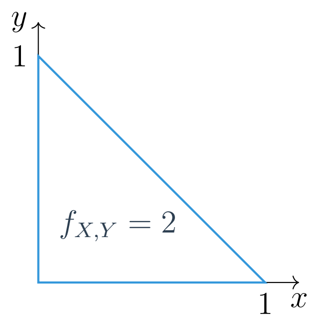
This is a classic "first joint distribution" exercise. Go slowly and always picture the triangle.

1. Joint pdf on a triangle

Suppose (X, Y) has joint pdf

$$f_{X,Y}(x, y) = \begin{cases} 2, & x \geq 0,\ y \geq 0,\ x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

So the support is the right triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$.



Why 2? The area of the triangle is

$$\text{area} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

To make the total probability 1, we need

$$\iint_{\text{triangle}} f_{X,Y}(x, y)\, dy\, dx = f \times \text{area} = 1 \quad \Rightarrow \quad f = \frac{1}{\text{area}} = \frac{1}{1/2} = 2.$$

So this constant is the *correct normalization*.

2. Joint CDF $F_{X,Y}(x, y) = P(X \le x, Y \le y)$

We must consider where (x, y) sits relative to the triangle.

Case 0: $x \le 0$ or $y \le 0$

Then there is no area in the first quadrant under (x, y), so

$$F_{X,Y}(x, y) = 0.$$

Case 1: $0 < x \le 1$, $0 < y \le 1$, and $x + y \le 1$

Here the rectangle $[0, x] \times [0, y]$ is completely inside the triangle (because any (u, v) in that rectangle satisfies $u \le x$, $v \le y$, so $u + v \le x + y \le 1$). So the probability is just density $\times$ area of the rectangle:

$$F_{X,Y}(x, y) = \int_0^x \int_0^y 2\,dv\,du = \int_0^x 2y\,du = 2xy.$$

So in this region:

$$F_{X,Y}(x, y) = 2xy.$$

Case 2: $0 < x \le 1$, $0 < y \le 1$, and $x + y > 1$

Now the rectangle $[0, x] \times [0, y]$ sticks out of the triangle above the line $u + v = 1$. We only integrate up to the line.

For a fixed u, the triangle allows v from 0 up to $1 - u$. But our rectangle wants v up to y. Since we are in the case $x + y > 1$, the line $u + v = 1$ is hit *before* we reach y, for u large enough. The split point is at $u = 1 - y$.

So:

$$F_{X,Y}(x, y) = \underbrace{\int_{u=0}^{1-y} \int_{v=0}^{y} 2\,dv\,du}_{\text{rectangle part}} + \underbrace{\int_{u=1-y}^{x} \int_{v=0}^{1-u} 2\,dv\,du}_{\text{triangular cap}}.$$

Compute the two pieces.

First piece.

$$\int_0^{1-y} \int_0^y 2\,dv\,du = \int_0^{1-y} 2y\,du = 2y(1 - y).$$

Second piece.

$$\int_{1-y}^{x} \int_0^{1-u} 2\,dv\,du = \int_{1-y}^{x} 2(1 - u)\,du = 2\int_{1-y}^{x} (1 - u)\,du.$$

Now

$$\int (1 - u)\,du = u - \frac{u^2}{2},$$

so

$$2\Big[u - \frac{u^2}{2}\Big]_{u=1-y}^{u=x} = 2\Big(x - \frac{x^2}{2} - (1-y) + \frac{(1-y)^2}{2}\Big).$$

Add both pieces:

$$F_{X,Y}(x,y) = 2y(1-y) + 2\Big(x - \frac{x^2}{2} - (1-y) + \frac{(1-y)^2}{2}\Big),$$

valid for $0 < x \leq 1$, $0 < y \leq 1$, $x + y > 1$.

You can leave it like that, or simplify. A clean simplified version is

$$F_{X,Y}(x,y) = 2x + 2y - x^2 - y^2 - 1, \qquad 0 < x \leq 1,\ 0 < y \leq 1,\ x + y > 1.$$

(You can check that on the boundary $x+y = 1$, both formulas give the same value $2x(1-x)$.)

Case 3: $x \geq 1$, $y \geq 1$

Then we have covered the whole triangle, so

$$F_{X,Y}(x,y) = 1.$$

Case 4: Mixed edges

If $x \geq 1$ but $0 < y < 1$, then we just clamp x to 1:

$$F_{X,Y}(x,y) = F_{X,Y}(1,y).$$

Similarly, if $y \geq 1$ but $0 < x < 1$, then

$$F_{X,Y}(x,y) = F_{X,Y}(x,1).$$

Final piecewise CDF:

$$F_{X,Y}(x,y) = \begin{cases} 0, & x \leq 0 \text{ or } y \leq 0, \\ 2xy, & 0 < x \leq 1,\ 0 < y \leq 1,\ x + y \leq 1, \\ 2x + 2y - x^2 - y^2 - 1, & 0 < x \leq 1,\ 0 < y \leq 1,\ x + y > 1, \\ F_{X,Y}(1,y), & x > 1,\ 0 < y \leq 1, \\ F_{X,Y}(x,1), & 0 < x \leq 1,\ y > 1, \\ 1, & x > 1,\ y > 1. \end{cases}$$

3. Recovering the pdf from the CDF

A good check: if we take

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{X,Y}(x,y),$$

do we get back 2 on the triangle?

- In the region $x > 0, y > 0, x + y < 1$, we had $F_{X,Y}(x,y) = 2xy$.

  $$\frac{\partial}{\partial x}(2xy) = 2y, \quad \frac{\partial}{\partial y}(2y) = 2.$$

  So $f_{X,Y}(x,y) = 2$ there.
- In the region $x > 0, y > 0, x+y > 1$ (but still in $[0,1]^2$), we had $F = 2x+2y-x^2-y^2-1$.

  $$\frac{\partial}{\partial x}F = 2 - 2x, \quad \frac{\partial}{\partial y}(2 - 2x) = 0,$$

  so $f_{X,Y} = 0$ there, as it should be (we are above the triangle).

So the CDF and pdf are consistent.

4. Marginal densities

Intro-to-prob style: once you trust the joint pdf, you should be able to get f_X and f_Y.

(a) Marginal of X

For a fixed $x \in (0,1)$, y can go from 0 up to $1-x$ inside the triangle.

$$f_X(x) = \int_0^{1-x} 2\,dy = 2(1-x), \quad 0 < x < 1,$$

and $f_X(x) = 0$ otherwise.

Check:

$$\int_0^1 2(1-x)\,dx = 2\left[x - \frac{x^2}{2}\right]_0^1 = 2\left(1 - \frac{1}{2}\right) = 1.$$

So f_X is a valid pdf.

(b) Marginal of Y

By symmetry,

$$f_Y(y) = \int_0^{1-y} 2\,dx = 2(1-y), \quad 0 < y < 1,$$

and $f_Y(y) = 0$ otherwise.

Again

$$\int_0^1 2(1-y)\,dy = 1,$$

so it's valid.

Takeaways

- Always make sure your density is well normalized.
- For $F_{X,Y}$, the only tricky case is when the rectangle crosses the line $x + y = 1$; split the integral there.
- Differentiating the CDF must give back the pdf.
- Marginals for this triangle come out linear: $f_X(x) = 2(1-x)$, $f_Y(y) = 2(1-y)$.