STAT 131 — Discussion (Lec 6 & Lec 7) Solutions

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How to use this

For each problem we **set up** the notation, give the **key idea**, and then **box** the final answer. Short "Teaching Notes" highlight recurring patterns.

Discussion (Lec 6)

1. Two-point discrete distribution written like a Bernoulli pmf

Let X take values a and b with $P(X=a)=p$ and $P(X=b)=1-p$.

Goal. Write the pmf in a compact Bernoulli-like way.

Solution. Using indicator notation,

$$p_X(x) = p\,\mathbf{1}\{x=a\} + (1-p)\,\mathbf{1}\{x=b\} = p^{\mathbf{1}\{x=a\}}(1-p)^{\mathbf{1}\{x=b\}}\,\mathbf{1}\{x\in\{a,b\}\}.$$

This matches the style $p^x(1-p)^{1-x}$ for $x\in\{0,1\}$ used for the Bernoulli.

$$\textbf{Final:}\quad \boxed{p_X(x) = p\,\mathbf{1}\{x=a\} + (1-p)\,\mathbf{1}\{x=b\}}.$$

Note

For two-point distributions, mapping $Y=\mathbf{1}\{X=a\}$ turns X into a Bernoulli(p), which is often convenient for algebra.

2. Two independent Poisson entrances, total cars in one hour

Entrance I: $N_1 \sim \text{Poisson}(\lambda_1 = 3)$; Entrance II: $N_2 \sim \text{Poisson}(\lambda_2 = 4)$, independent. Total $N = N_1 + N_2$.

Key fact. Sum of independent Poisson random variables is Poisson with rate $\lambda_1 + \lambda_2$.

Hence $N \sim \text{Poisson}(7)$ and

$$P(N = 3) = e^{-7}\frac{7^3}{3!} = \boxed{e^{-7}\frac{343}{6} \approx 0.0522}.$$

Teaching Note

If you ever forget the sum-of-Poissons rule, you can convolve pmfs directly; the result simplifies to a Poisson again.

Discussion (Lec 7)

1. Polynomial pdf on a finite interval

Given

$$f(x) = \begin{cases} c\,(9 - x^2), & -3 \le x \le 3, \\ 0, & \text{otherwise}, \end{cases} \qquad c > 0.$$

(a) Find c and describe the shape. Normalization:

$$1 = \int_{-3}^{3} c(9 - x^2)\,\mathrm{d}x = c\Big[9x - \frac{x^3}{3}\Big]_{-3}^{3} = c\big((27-9) - (-27+9)\big) = 36c.$$

Thus $c = \boxed{\frac{1}{36}}$. The graph is an even (symmetric) downward-opening parabola on $[-3, 3]$, touching zero at ± 3.

(b) Compute $P(X < 0)$, $P(-1 \le X \le 1)$, $P(X > 2)$. Because f is even and supported symmetrically,

$$P(X < 0) = \tfrac{1}{2} = \boxed{\tfrac{1}{2}}.$$

Next,

$$P(-1 \le X \le 1) = \int_{-1}^{1} \frac{1}{36}(9 - x^2)\,\mathrm{d}x = \frac{1}{36}\Big[9x - \frac{x^3}{3}\Big]_{-1}^{1} = \frac{1}{36}\cdot\frac{52}{3} = \boxed{\frac{13}{27} \approx 0.4815}.$$

Finally,

$$P(X > 2) = \int_{2}^{3} \frac{1}{36}(9 - x^2)\,\mathrm{d}x = \frac{1}{36}\Big[9x - \frac{x^3}{3}\Big]_{2}^{3} = \frac{1}{36}\cdot\frac{8}{3} = \boxed{\frac{2}{27} \approx 0.0741}.$$

(Consistency: $P(|X| > 2) = 2 \cdot \frac{2}{27} = \frac{4}{27}$.)

Note

When the pdf is even on $[-a, a]$, immediately record $P(X < 0) = \frac{1}{2}$ and convert any symmetric integral to twice the integral over $[0, \cdot]$.

2. Exponential cdf

Let X be exponential with parameter $\beta > 0$. (Here we use the *rate* parameterization: $f(x) = \beta e^{-\beta x}$ for $x \geq 0$; note that some texts use β as a *scale* so $f(x) = \frac{1}{\beta}e^{-x/\beta}$. The derivation is identical with $\beta \mapsto 1/\beta$.)

Derivation. For $x < 0$, $F_X(x) = P(X \leq x) = 0$. For $x \geq 0$,

$$F_X(x) = \int_0^x \beta e^{-\beta t}\, \mathrm{d}t = \Big[-e^{-\beta t}\Big]_0^x = 1 - e^{-\beta x}.$$

$$\textbf{Final cdf:} \quad \boxed{F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\beta x}, & x \geq 0. \end{cases}}$$

Note (memoryless)

For $X \sim \mathrm{Exp}(\beta)$ (rate), $P(X > s + t \mid X > s) = P(X > t)$; this property often unlocks queueing/waiting-time problems quickly.