STAT 131 — Discussion 0: Math Refresher (Solutions)

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Scope

This handout solves every item in the *Discussion/Refresher* section only. Homework problems are intentionally omitted. No figures or sketches are used.

Glossary (used consistently)

- **Integration by Parts (IBP):** $\displaystyle\int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u.$
- **Fundamental Theorem of Calculus (FTC):** $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{g(x)} f(t)\,\mathrm{d}t\right) = f(g(x))\,g'(x)$ (FTC + Chain Rule).
- **Convexity:** f is convex if $f''(x) \geq 0$; its graph lies above every tangent line.
- **Swap order of integration:** Re-express $\iint_R (\cdot)\,\mathrm{d}x\,\mathrm{d}y$ by carefully rewriting the region R in the opposite order.

0.1 Math Refresher — Fully Worked

A) Series: definitions and simplifications (Example 0.1)

Let $x \in \mathbb{R}$. We derive each closed form, keeping steps explicit.

(a) Finite geometric sum. **Claim.** $\displaystyle\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for $x \neq 1$ (and $= n+1$ when $x = 1$).

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$ and suppose $x \neq 1$. Then

$$\begin{aligned}
xS_n &= x + x^2 + \cdots + x^{n+1}, \\
S_n - xS_n &= (1 + x + \cdots + x^n) - (x + \cdots + x^{n+1}) = 1 - x^{n+1}, \\
(1-x)S_n &= 1 - x^{n+1} \quad \Rightarrow \quad S_n = \frac{1-x^{n+1}}{1-x}.
\end{aligned}$$

If $x = 1$, the sum is $n + 1$. □

(b) Infinite geometric sum ($|x| < 1$). **Claim.** $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$.

Proof. Take $n \to \infty$ in part (a). Since $|x| < 1$, we have $x^{n+1} \to 0$, hence

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}. \ \square$$

(c) Arithmetic–geometric sums (finite and infinite). **Finite.** For $x \neq 1$,

$$\begin{aligned}
\sum_{k=1}^{n} k\, x^{k-1} &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(\sum_{k=0}^{n} x^k\Big) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1-x^{n+1}}{1-x}\Big) \\
&= \frac{\big(-(n+1)x^n\big)(1-x) - (1-x^{n+1})(-1)}{(1-x)^2} \quad \text{(quotient rule)} \\
&= \frac{-(n+1)x^n + (n+1)x^{n+1} + 1 - x^{n+1}}{(1-x)^2} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.
\end{aligned}$$

Multiplying by x gives

$$\sum_{k=1}^{n} k\, x^k = \frac{x\big(1-(n+1)x^n + nx^{n+1}\big)}{(1-x)^2}.$$

Infinite ($|x| < 1$). Letting $n \to \infty$ (so $x^n \to 0$ and $x^{n+1} \to 0$) yields

$$\sum_{k=1}^{\infty} k\, x^{k-1} = \frac{1}{(1-x)^2}, \qquad \sum_{k=1}^{\infty} k\, x^k = \frac{x}{(1-x)^2}. \ \square$$

B) Integrals and changing order (Example 0.2)

1. Single integral. Evaluate $\int_0^\infty y\,e^{-y}\,\mathrm{d}y$. **Solution (IBP).** Take $u = y$ and $\mathrm{d}v = e^{-y}\,\mathrm{d}y$; then $\mathrm{d}u = \mathrm{d}y$ and $v = -e^{-y}$.

$$\int_0^\infty y\,e^{-y}\,\mathrm{d}y = \Big[-y\,e^{-y}\Big]_0^\infty + \int_0^\infty e^{-y}\,\mathrm{d}y.$$

We have $\lim_{y\to\infty} ye^{-y} = 0$ (e.g., by L'Hôpital: $\lim_{y\to\infty} y/e^y = \lim 1/e^y = 0$), so the boundary term is $0 - 0 = 0$ and

$$\int_0^\infty e^{-y}\,\mathrm{d}y = \Big[-e^{-y}\Big]_0^\infty = 1.$$

Therefore $\int_0^\infty y\,e^{-y}\,\mathrm{d}y = \boxed{1}$. $\square$

2. Triangular double integral with e^{-y}. Evaluate $\int_0^\infty\int_0^y e^{-y}\,\mathrm{d}x\,\mathrm{d}y$. **Solution.** The inner integral does not depend on x, so

$$\int_0^y e^{-y}\,\mathrm{d}x = (y-0)\,e^{-y} = y\,e^{-y}.$$

Thus

$$\int_0^\infty\int_0^y e^{-y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty y\,e^{-y}\,\mathrm{d}y = \boxed{1}.$$

(Alternatively, swapping order gives the same value.) $\square$

3. Log-shaped region. Evaluate $\int_{x=1}^{e}\int_{y=0}^{\log x} 1\,\mathrm{d}y\,\mathrm{d}x$. **Solution (as written).**

$$\int_1^e \Big[\int_0^{\log x} 1\,\mathrm{d}y\Big]\,\mathrm{d}x = \int_1^e \log x\,\mathrm{d}x = \big[x\log x - x\big]_1^e = \boxed{1}.$$

Same value by swapping order. The region satisfies $1 \le x \le e$ and $0 \le y \le \log x$. Equivalently: $0 \le y \le 1$ and $1 \le x \le e^y$, so

$$\int_{y=0}^{1}\int_{x=1}^{e^y} 1\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 (e^y - 1)\,\mathrm{d}y = \big[e^y - y\big]_0^1 = \boxed{1}. \ \square$$

C) Fundamental Theorem of Calculus (FTC)

If f is continuous on $[a,b]$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{g(x)} f(t)\,\mathrm{d}t\right) = f(g(x))\,g'(x).$$

Plain-English version

Differentiate an "area-so-far" function by evaluating the integrand at the moving limit, then multiply by the speed of that limit (the Chain Rule).

D) Euler's number e (two limits; simple rigorous proofs)

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Proof. Let $a_n = (1+\frac{1}{n})^n$. Then $\ln a_n = n\ln(1+\frac{1}{n})$. Use the elementary bounds (valid for $u > -1$):

$$-\frac{u^2}{2} \;\le\; \ln(1+u) - u \;\le\; 0.$$

With $u = \frac{1}{n}$, we get

$$1 - \frac{1}{2n} \;\le\; n\ln\Big(1+\tfrac{1}{n}\Big) \;\le\; 1.$$

By the squeeze theorem, $n\ln(1+\frac{1}{n}) \to 1$, hence $\ln a_n \to 1$ and $a_n \to e$. $\square$

2. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for fixed $x\in\mathbb{R}$.

Proof. Let $b_n = (1+\frac{x}{n})^n$. Then $\ln b_n = n\ln(1+\frac{x}{n})$ and, with $u = \frac{x}{n}$,

$$-\frac{x^2}{2n} \;\le\; n\ln\Big(1+\tfrac{x}{n}\Big) - x \;\le\; 0.$$

Thus $n\ln(1+\frac{x}{n}) \to x$, so $\ln b_n \to x$ and $b_n \to e^x$. $\square$

E) Comparing e^x with lines on $(0,1)$ (Example 0.3)

(a) $e^x > 1+x$ for $0<x<1$. **Proof 1 (series).** $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots > 1+x$.
Proof 2 (convexity). $f(x) = e^x$ has $f''(x) = e^x > 0$, so f is convex. The tangent line at 0 is $1+x$, and a convex function lies above its tangents, with equality only at the point of tangency. Hence $e^x > 1+x$ for $x \neq 0$. $\square$

(b) $e^{-x} > 1-x$ for $0<x<1$. **Proof 1 (series).** $e^{-x} = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{3!} + \cdots > 1-x$.
Proof 2 (calculus). Let $g(x) = e^{-x} - (1-x)$. Then $g(0) = 0$ and $g'(x) = -e^{-x} + 1 = 1 - e^{-x} \ge 0$ for $x \ge 0$. Hence g is increasing on $[0,\infty)$ and $g(x) > 0$ for $x > 0$. $\square$

F) Harmonic number size and bounds

Define $H_n = \sum_{k=1}^n \frac{1}{k}$. We give the standard integral comparison.

Claim. For $n \geq 1$,

$$\log(n+1) \;\leq\; H_{n+1} \;\leq\; 1 + \log n.$$

Proof. For $x \in [k, k+1]$ with integer $k \geq 1$, we have $\dfrac{1}{k+1} \leq \dfrac{1}{x} \leq \dfrac{1}{k}$. Integrating over $[k, k+1]$ and summing $k = 1, \ldots, n$ yields

$$\sum_{k=1}^{n} \frac{1}{k+1} \;\leq\; \int_{1}^{n+1} \frac{\mathrm{d}x}{x} \;\leq\; \sum_{k=1}^{n} \frac{1}{k}.$$

The left sum is $H_{n+1} - 1$; the right sum is H_n. Rearranging gives the stated bounds. □

Tiny self-checks

1. Differentiate $\sum_{k=0}^{\infty} x^k$ (for $|x| < 1$) to confirm $\sum_{k=0}^{\infty} kx^{k-1} = \dfrac{1}{(1-x)^2}$.
2. Swap the order of $\int_0^2 \int_0^y (y+1)\,\mathrm{d}x\,\mathrm{d}y$ and recompute (you should get $\int_0^2 (2y - y^2)\,\mathrm{d}y$).
3. Use convexity to give a one-line proof that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

If a step feels too fast, circle it and ask about that exact line. Specific questions help everyone.